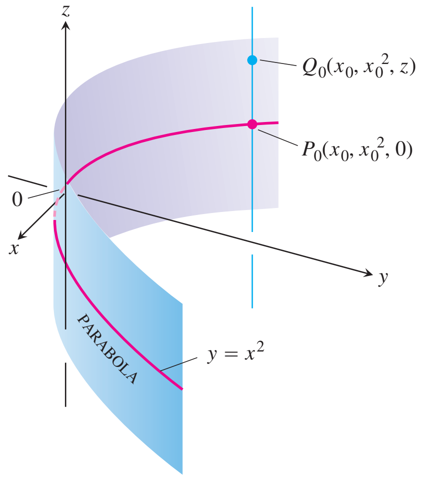


FIGURE 12.44 Every point of the cylinder in Example 1 has coordinates of the form $(x_0, x_0{}^2, z)$. We call it "the cylinder $y = x^2$."

As Example 1 suggests, any curve $f(x, y) = c$ in the xy-plane defines a cylinder parallel to the z-axis whose equation is also $f(x, y) = c$. For instance, the equation $x^2 + y^2 = 1$ defines the circular cylinder made by the lines parallel to the z-axis that pass through the circle $x^2 + y^2 = 1$ in the xy-plane.

In a similar way, any curve $g(x, z) = c$ in the xz-plane defines a cylinder parallel to the y-axis whose space equation is also $g(x, z) = c$. Any curve $h(y, z) = c$ defines a cylinder parallel to the x-axis whose space equation is also $h(y, z) = c$. The axis of a cylinder need not be parallel to a coordinate axis, however.

Quadric Surfaces

A **quadric surface** is the graph in space of a second-degree equation in x, y, and z. We focus on the special equation

$$Ax^2 + By^2 + Cz^2 + Dz = E,$$

where A, B, C, D, and E are constants. The basic quadric surfaces are **ellipsoids**, **paraboloids**, **elliptical cones**, and **hyperboloids**. Spheres are special cases of ellipsoids. We present a few examples illustrating how to sketch a quadric surface, and then give a summary table of graphs of the basic types.

EXAMPLE 2 The **ellipsoid**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(Figure 12.45) cuts the coordinate axes at $(\pm a, 0, 0)$, $(0, \pm b, 0)$, and $(0, 0, \pm c)$. It lies within the rectangular box defined by the inequalities $|x| \leq a$, $|y| \leq b$, and $|z| \leq c$. The surface is symmetric with respect to each of the coordinate planes because each variable in the defining equation is squared.

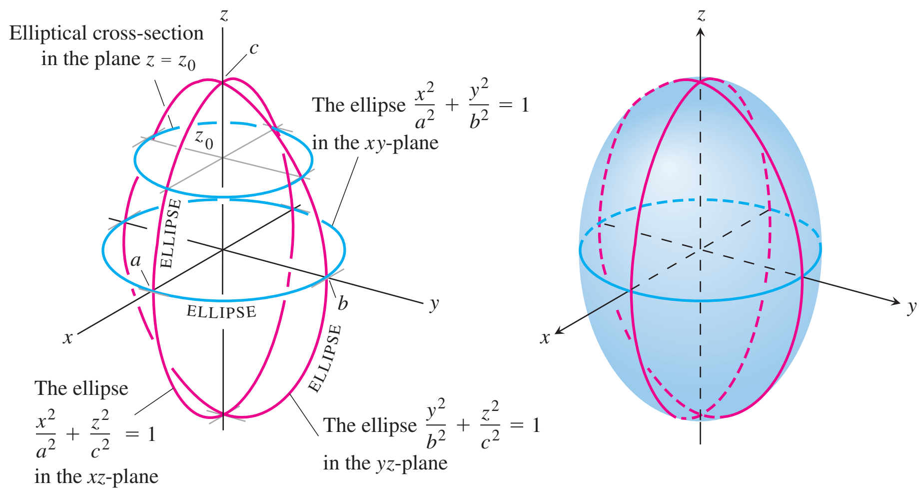


FIGURE 12.45 The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in Example 2 has elliptical cross-sections in each of the three coordinate planes.

The curves in which the three coordinate planes cut the surface are ellipses. For example,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{when} \qquad z = 0.$$

The curve cut from the surface by the plane $z = z_0$, $|z_0| < c$, is the ellipse

$$\frac{x^2}{a^2(1 - (z_0/c)^2)} + \frac{y^2}{b^2(1 - (z_0/c)^2)} = 1.$$

If any two of the semiaxes a, b, and c are equal, the surface is an **ellipsoid of revolution**. If all three are equal, the surface is a sphere. ■

EXAMPLE 3 The **hyperbolic paraboloid**

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \qquad c > 0$$

has symmetry with respect to the planes $x = 0$ and $y = 0$ (Figure 12.46). The cross-sections in these planes are

$$x = 0: \quad \text{the parabola } z = \frac{c}{b^2}y^2. \tag{1}$$

$$y = 0: \quad \text{the parabola } z = -\frac{c}{a^2}x^2. \tag{2}$$

In the plane $x = 0$, the parabola opens upward from the origin. The parabola in the plane $y = 0$ opens downward.

If we cut the surface by a plane $z = z_0 > 0$, the cross-section is a hyperbola,

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z_0}{c},$$

with its focal axis parallel to the y-axis and its vertices on the parabola in Equation (1). If z_0 is negative, the focal axis is parallel to the x-axis and the vertices lie on the parabola in Equation (2).

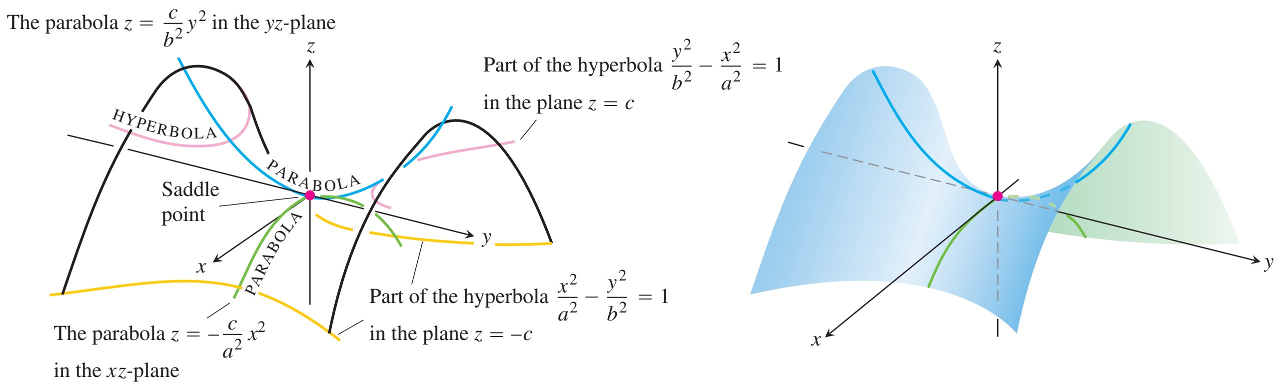
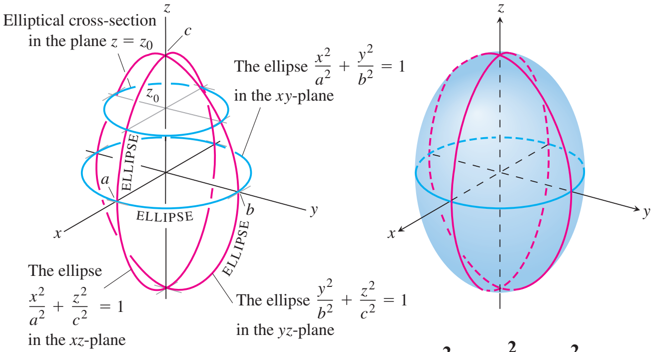


FIGURE 12.46 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c$, $c > 0$. The cross-sections in planes perpendicular to the z-axis above and below the xy-plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

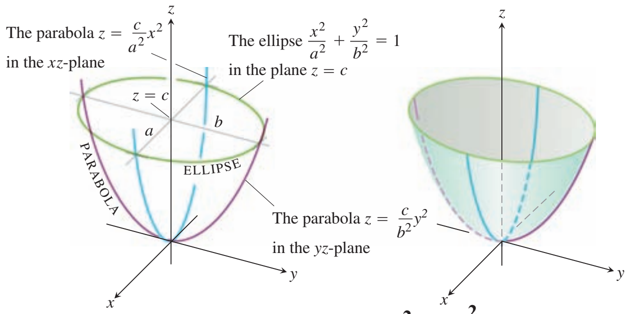
Near the origin, the surface is shaped like a saddle or mountain pass. To a person traveling along the surface in the yz-plane the origin looks like a minimum. To a person traveling the xz-plane the origin looks like a maximum. Such a point is called a **saddle point** of a surface. We will say more about saddle points in Section 14.7. ■

Table 12.1 shows graphs of the six basic types of quadric surfaces. Each surface shown is symmetric with respect to the z-axis, but other coordinate axes can serve as well (with appropriate changes to the equation).

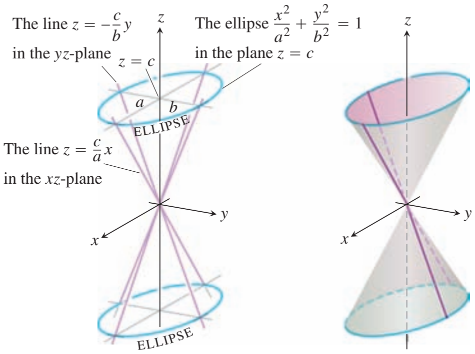
TABLE 12.1 Graphs of Quadric Surfaces



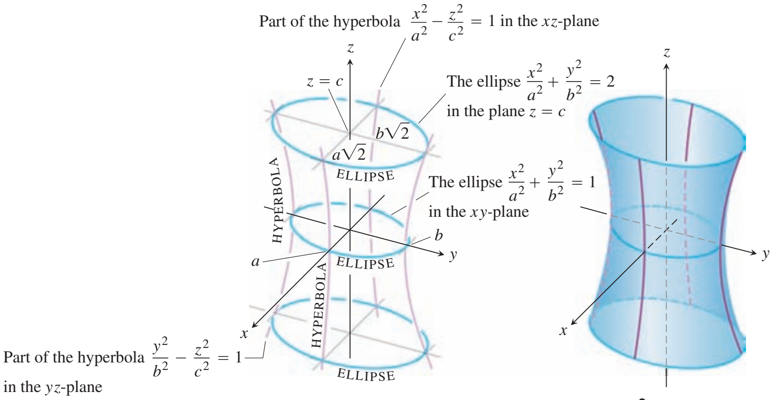
ELLIPSOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



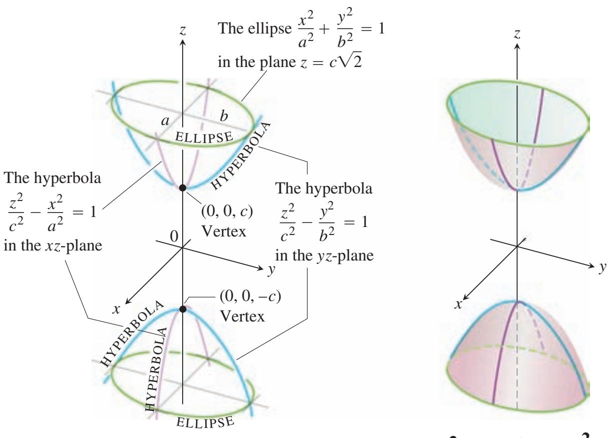
ELLIPTICAL PARABOLOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



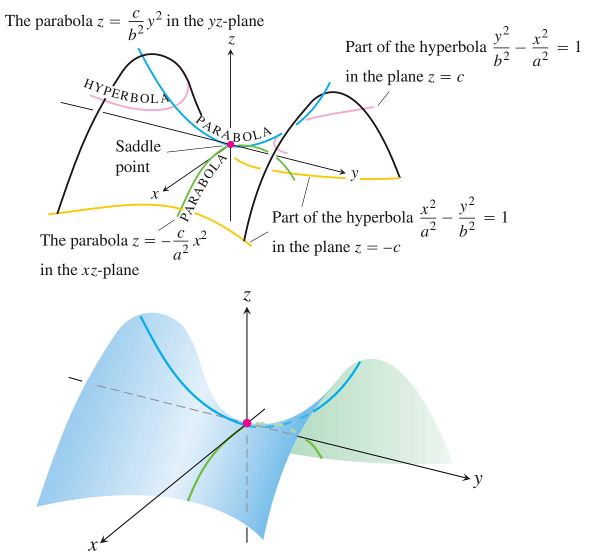
ELLIPTICAL CONE $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\quad \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$